Four foundational questions in probability theory and statistics

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Abstract: This study has the purpose of addressing four questions that lie at the base of the probability theory and statistics and includes two main steps. In the preliminary step, we conduct the textual analysis of the most significant works written by eminent probability theorists. The textual analysis turns out to be a rather innovative method of study in this domain and shows how the sampled writers—no matter whether a frequentist or a subjectivist—share a similar approach. Each author argues on the multifold aspects of probability, and then, he establishes the mathematical theory on the base of his intellectual conclusions. It may be said that mathematics ranks second. In the second stage of the present research, we address the four questions mentioned above using a purely mathematical approach instead of the way followed by the surveyed authors. This approach is not new, as Hilbert proposed to axiomatize the probability calculus; notably, he recommended to describe the probability concepts exclusively on the basis of mathematical criteria. In particular, we use two theorems that prove how the frequentist and the subjectivist models are not incompatible as many believe. Probability has distinct meanings under different hypotheses, and in turn, classical statistics and Bayesian statistics are available for adoption in different circumstances. Subsequently, these original conclusions are commented upon, followed by our conclusions. © *2017 Physics Essays Publication*. [http://dx.doi.org/10.4006/0836-1398-30.3.314]

Résumé: Cette étude vise à répondre à quatre questions qui se situent à la base de la théorie des probabilités et de la statistique, et comprend deux étapes principales. D'abord nous avons effectué une analyse textuelle des œuvres les plus significatives écrites par éminents théoriciens de probabilité. L'analyse textuelle est une méthode d'étude plutôt innovante dans ce domaine, et montre comment ces auteurs - peu importe qu'ils soient fréquentistes ou subjectivistes - partagent une approche similaire. Chaque auteur se penche sur les divers aspects de la probabilité puis il établit sa théorie mathématique sur la base de considérations personnelles. On peut dire que les mathématiques se classent en secondaire lieu. Dans la deuxième étape de la présente recherche, nous abordons les quatre questions mentionnées ci-dessus en utilisant une approche purement mathématique au lieu de la méthode suivie par les auteurs interrogés. Cette approche n'est pas nouvelle puisque Hilbert a proposé l'axiomatisation du calcul de probabilité, c'est à dire il a recommandé de décrire les concepts de probabilité sur la base des critères purement mathématiques tandis que les réflexions philosophiques ont un rôle auxiliaire. Plus précisément, nous utilisons deux théorèmes qui prouvent que les modèles fréquentistes et subjectivistes ne sont pas incompatibles comme l'on le croit habituellement. Les théorèmes montrent come la notion de probabilité a des significations distinctes sous différentes hypothèses. Par la suite les statistiques classiques et les statistiques bayésiennes peuvent être adoptés dans différentes circonstances. Ces conclusions originales sont commentées et sont suivis de nos remarques finales.

Key words: Foundational Issues; Frequentist and Subjective Probabilities; Classical and Bayesian Statistics.

I. INTRODUCTION

Pascal inaugurated the modern calculus of probability in 1654. Bernoulli, De Moivre, Leibniz, and others addressed complex problems by providing complete solutions, but the first attempt to define the concept of probability was authored by Laplace in 1812, over one century and half after Pascal. The Laplacian definition had nontrivial weak points; statistics began to gain weight in science and economics during the 19th century and many people, whether erudite or illiterate, felt the need for the precise description of the probability concept. Significant attempts have been made to answer this issue, but theorists progressed slowly.[1] We make the historical list of the constructions that the current literature recognizes as the most used ones:[2]

- The *frequentist* theory by von Mises in 1928
- The *subjective* theory by Ramsey and de Finetti (independently) in 1931
- The *axiomatic* theory by Kolmogorov in 1933
- The *Bayesian* theory by Savage in 1954
- The *logical* theory by Carnap in 1962.

We mean to focus on the most popular models of probability, the *frequentist* and the *subjective*, as long as authors

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agree in merging the subjective and Bayesian views into a single box. Frequentists define probability as the limit of the relative frequency in a large number of trials. Subjectivists see probability as an individual person's measure of belief that an event will occur. These interpretations show apparent incongruities, and the concept of probability still emerges as a conundrum three hundred and fifty years after Pascal.

II. TEXTUAL ANALYSIS

The dispute raised between frequentist and subjective schools attracted the attention of several theorists. Papers and books have been filled with annotations and sharp comments about this complex argument. We share the same interest but decided to take a different direction. Instead of formulating remarks and explanations, we have conducted the textual analysis of the principal *book* (B) or *essay* (E) prepared by each one of the following seven writers: Venn, von Mises, Reichenbach, Keynes, Ramsey, de Finetti, and Savage (Table I). We have selected these authors because they are recognized as the founders of the frequentist and subjective schools. We have overlooked other researchers in the fields who propose interesting solutions but in substances belonging to one of the two schools. For example, Cox defines probability as a measure of a degree of belief that is consistent with Boolean logic, and Knuth further generalizes this scheme to include other algebras and hence other problems in science and mathematics. However, Cox and Knuth take a tacit step in advance since they adhere to the subjectivist circle.

Experts of probability are not so familiar with textual analysis, and this is the first attempt to break down texts of probability into their components to the best of our knowledge. Textual analysis is not a bibliographical analysis. The latter usually consists in examining the largest number of works dealing with a certain topic; it can be catalogued as an intellectual and subjective survey. The former focuses on a predefined set of works and requires the researcher to closely investigate the objective content of each work;[11] it can be seen as a statistical and objective inquiry. For example, a researcher of textual analysis can count the number of times certain phrases or words that are used in the text; he can define the structure of the text on the basis of the chapters and the sections that compose it; he can dissect author's narrative technique, etc.

Let us see the principal outcomes from the textual analysis conducted on the works mentioned above.

(i) Some sections of the sampled texts are labeled as follows: "A problem of terminology" (Ref. 4, p. 93), "The nihilists" (Ref. 4, p. 97), 'The world and the state of the world' (Ref. 5, p. 8), "The value of observation" (Ref. 5, p. 125), "Distinction between logical and psychological view" (Ref. 7, p. 129), "The application of probability to conduct" (Ref. 8, p. 351), and "Tyranny of language" (Ref. 9, p. 28). The titles reveal the concern of writers about qualitative and philosophical themes. The writers argue over on a variety of topics which are distant from mathematics.

(ii) The authors fill the pages with comments, reflections, notes, explanations, and critical remarks; they share the verbose style of humanists. We use the book of Kolmogorov[3] as a comparison term to qualify this aspect of the sample. In fact, Kolmogorov's book does not devote any space to intellectual ruminations and does not make remarks about other views of probability. The percentage increase of pages related to the Kolmogorov book varies from +191% to +815%. The percentage increase cannot be calculated for Ramsey who prepared an essay and not a book.

(iii) Each master means to assess his view as the authentic and unique model of probability but does not demonstrates this statement using a mathematical proof. He disseminates critical annotations against the concurrent studies in the book and even gathers disapproving remarks in special places of the book. The right side of Table I exhibits the details of this criticism, while Appendix A shows a very short summary of the contents. It may be said that Appendix A expands each line of Table I (right side). The rightmost column shows the percentage extent of criticism with respect to the overall work extent and an idea of the authors' polemist efforts.

(iv) Basically, the works have this logical structure: the writer argues about the multifold nature of probability and other arguments too [see point (i)]. He criticizes some models and picks up the model of probability that he judges to be the best in accordance with his proper criteria, and finally, the author sets up the mathematical construct about the preferred interpretation of P. When

TABLE I. Textual features of the surveyed works.

	Work				Criticism			
	Author	Reference	Type	Extent W (Pages)	Chapter ID	Location(Initial and final pages)	Extent C(Pages)	C/W
	Kolmogorov	3	B	84	None	N.A.	0	N.A.
1	Von Mises	4	B	245 (+191%)	3	66–86	20	8.1
2	Savage	5	B	309 (+267%)	4	60–62	3	0.9
3	Reichenbach	6	B	489 (+482%)	9	366–386	21	4.2
4	Venn	7	B	508 (+504%)	6 and 10	119–166 and 235–264	78	15.3
5	Keynes	8	B	539 (+541%)	7 and 8	102–124	23	4.2
6	De Finetti	9	B	769 (+815%)	1, 6, and 12	10–15, 270–284, and 614–626;	34	4.4
7	Ramsey	10	E	43 (N.A.)	1 and 2	158–166	8	18.6

two authors share a common view, they can even reach opposite conclusions because of their personal approach. For example, Reichenbach (Ref. 6, p. 13) and de Finetti[12] agree that the probability of a single event has no relation with the physical world. The former concludes that this form of probability is to be rejected because of its unreality, and the latter places it at the core of his construction.

In conclusion, the textual analysis brings evidence on how the sampled authors—no matter whether frequentists or subjectivists—share a similar style which is strongly based on intellectual discussion, while mathematics ranks second. This method of study did not solve the foundational issues of probability and statistics in a definitive manner.

III. OPEN PROBLEMS

Let us recall four significant questions that are still open.

A. The problem of interpretation

The probability calculus is capable of solving intricate problems, and the statistical methods offer support to address sophisticated previsions. Sometimes, an expert obtains the value of probability P at the end of admirable efforts, but he is unable to explain whether that number qualifies a material fact or expresses personal credence in that fact. He cannot ensure whether the number P is a chance, a possibility, or a wish, and the *interpretation problem* is still a debated argument as we have recalled in the introduction of this paper. This fault turns out to be not negligible since statistics has infiltrated several sectors of the present global society and people exploit probability calculations in various areas of modern economies.

B. The problem of discrepancy

The frequentist and subjective models underpin the classical and Bayesian statistics in a certain way,[13] and therefore, statistical applications should falsify one of the two theories. If the probability models were irreconcilable, as some masters hold, the use of statistics should lend support to one model and should prove that the other is false. Instead, countless professional cases demonstrate that both classical and Bayesian statistics are correct, and sometimes, they furnish identical results. We highlight that a blatant contradiction emerges between theory and practice in the probability domain, and this *discrepancy problem* is awaiting an answer.

C. The problem of choice

Classical and Bayesian statistics are normally used in professional practice, and sometimes, working statisticians wonder: What is the most appropriate statistics to be employed in a scheduled project? There is a certain interplay between the two statistics, and a precise criterion for selecting the better is missing. An investor who pays for a statistical study and requests for the optimal procedure to follow does not obtain a precise answer. The rigorous rule to decide on the most appropriate way forward for a planned project is lacking.[14–16] Two statisticians may well disagree about the most suitable statistics for given prerequisites, and the *choice problem* is still debated.

D. The problem of axiomatization

In the year 1900, David Hilbert illustrated a list of problems, unsolved at that time, at the International Congress of Mathematicians held in Paris. The sixth statement of Hilbert suggested the formal exposition of the axioms of the probability calculus.[17] During the 20th century, the studies on probability progressed, and now, we have different axiomatizations in the literature. There are also several "pluralist" writers who accept more than one interpretation;[18] we mention Jean–Antoine Condorcet, Joseph Louis Bertrand, Henry Poincaré, Antoine Cournot, Denis Poisson, Bernard Bolzano, Robert Leslie Ellis, Jacob Friedrich Fries, and Karl Popper. Faced with such a plethora of eminent pluralist authors, one might wonder whether Hilbert's sixth problem has several solutions instead of only one; hence, one wonders: Is the *axiomatization problem* an ill posed question? The doubts about Hilbert's sixth problem are still in force.

At present, science progresses at high speed; but after three centuries, significant theoretical and practical issues continue to lie at the basis of the probability calculus and statistics.

IV. THE MATHEMATICAL APPROACH TO DEFINE A MEASURE

Mathematics ranks second in the method adopted by the surveyed authors; instead, we mean to employ a method where mathematics ranks first. This is not new. Hilbert in his famous paper wrote: "The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics." Hilbert encouraged theorists to establish the fundamental aspects of probability without any other tool than mathematics. Kolmogorov applies this approach[19] and neither starts with philosophical considerations nor makes remarks about other views of probability. Kolmogorov's book presents six axioms and treats purely mathematical topics such as infinite probability fields, random variables, mathematical expectations, conditional probability, and independence. We share this approach guided by the mathematical logic since the efforts analyzed in Sec. II left the *Problem of Interpretation* unresolved and in addition have raised or aggravated the *Problems of Discrepancy*, of *Choice* and *Axiomatization*. We decided to address the questions from a to d using the approach which is recognized as essentially mathematical.

It is necessary to specify that the attribute "mathematical" pertains to the mode employed by all the physicists, engineers, and researchers who determine the new parameter or new measure y using mathematical tools, while verbal comments and remarks have a secondary position. Obviously, explanations are to be added to formal expressions, but equations and theorems rank first. This method is a typical part of science theorization and centers on the analytical equation that determines y in any aspect—e.g., $y = f(x)$—more precisely, the *argument* or *free variable* x yields the numerical value and also the significance of y. The measure y can have the *general-abstract* meaning and various *specific-practical* meanings, and this grouping corresponds to the usual subdivision of the mathematical calculus which includes two areas with different properties and scopes.[20] On one side, pure mathematicians treat topics that have a generic relationship with physical reality; on the other side, technicians, physicists, economists, and other professionals treat topics that have specific relationships with practical applications.

Our personal efforts to apply the formal method have a long history. We wrote a book[21] which discusses the importance and roles that the probability argument has from the theoretical viewpoint. An article[22] published in *Physics Essays* summarizes these initial results. Later, the book[23] has calculated the frequentist and subjective probabilities using two theorems. This part of the inquiry will be summarized in the following pages.

V. ADVANTAGES OF THE MATHEMATICAL APPROACH IN THE PROBABILITY DOMAIN

The mathematical approach needs the variable x to fix y, and so, it is necessary to establish the argument in advance of defining the probability. Pascal inaugurated the calculus of chance in around 1654, but the precise argument of P remained undefined until 1933 when Kolmogorov first fixed it in formal terms.[24] At present, broad literature recognizes that *the random event is the argument of probability*, while probability is *the measure of how likely an event will happen*

$$P = P(\text{random event}). \tag{1}$$

Kolmogorov defines the random event apart from concrete existence and establishes that E is a *subset* of event space $\mathcal{F}$

$$P = P(E), \quad E \in \mathcal{F}; \quad P \in \mathbb{R} \tag{2}$$

Thus, $P(E)$ has *general significance;* it is *the probability in abstract* typical of the axiomatic theory.

In applications, there are two noteworthy arguments that are *the long-term event* E_n—also called *collective* by von Mises and *series* by Venn—and *the single event* E_1. We look into the properties of $P(E_n)$ and $P(E_1)$ using a pair of theorems; in particular, the theorems—demonstrated in Ref. 23—illustrate the relationships that exist between the relative frequency and the probabilities $P(E_n)$ and $P(E_1)$ in the order.

RESULT 1

It is assumed the Bernoulli scheme and that the concept of probability refer systematically to the probability of success. The former is the well-known *Theorem* or *Law of Large Numbers* (TLN), which in the strong form can be expressed in the following way

$$F(E_n) \overset{a.s.}{\rightarrow} P(E_n), \qquad \text{as } n \rightarrow \infty. \tag{3}$$

The *Theorem of a Single Number* (TSN) holds that the relative frequency of success is not equal to the probability in a single trial

$$F(E_1) \neq P(E_1), \quad n = 1. \tag{4}$$

In other words, the frequency gets close to the probability of the long-term event $P(E_n)$ when n tends to infinity, and the frequency does not match with the probability of the single event $P(E_1)$. Appendix B includes a lemma that completes the second result and establishes the upper bound for the property (4). For the sake of simplicity, this paper focuses on the single event instead of the set of events defined by the lemma.

Speaking in general, it is the job of researchers to collect plausible explanations and to use scientific experiments to filter through them, retaining ideas that are supported by the evidence and discarding the others. Testing is at the core of the scientific method. If experiments do not fit with y, or y is not testable in absolute, then researchers conclude that the parameter y is not extant in the physical world; it has no actual substance.

TLN and TSN explain how P can be validated in the physical reality. Specifically, TLN demonstrates that—at least in principle—the probability of repeated events can be tested, and hence, $P(E_n)$ is a parameter that exists in the world, it is *an authentic physical quantity*. In addition, assumption $n \rightarrow \infty$ is consistent with the classical statistical inference that makes propositions about a population, using data drawn from the population with some form of sampling. This is the first part of the answer to *Problem A,* and the second part is more complex.

TSN proves that one cannot test the probability of a single random event. It is not a question of tools or environmental constraints; the theorem proves that never and ever one can control $P(E_1)$, and hence, $P(E_1)$ *does not qualify a physical quantity*. This conclusion drawn from TSN fits perfectly with the famous aphorism of de Finetti, "Probability does not exist," which has raised much discussion. Some commentators object that this aphorism sounds like a "radical" statement, others judge it inappropriate for applications, and so on. In fact, when one expresses the personal opinion X, others have a legitimate right to contradict it. Instead, when a theorem proves X, either one disproves the theorem or private judgments are not allowed about X.

As a consequence of TSN, the probability of a single trial must be discarded from the scientific realm and scholars must refuse to accept it.

RESULT 2

Now, a contradiction seems to emerge between the TSN and the professional needs: scientists should reject $P(E_1)$, but instead, they are called for the calculation of $P(E_1)$ every day. How this apparent discrepancy between theory and practice is justified?

The answer may be found in the semiotic studies. *Semiotics* teaches us that words and numbers are *signifiers*—usually called *pieces of information*—that, by definition, stand for something.[25] For example, the number 0.32 *Km* represents a physical distance, and also the number 0.32 [$=P(E_n)$] signifies a physical quantity. The number 0.32 [$=P(E_1)$] does not represent a physical quantity but is still an item of information. The probability of a single trial has semantic value since it is a meaning conveyor. As a consequence of the informational position held by $P(E_1)$, theorists and professionals are allowed to recycle $P(E_1)$ that they should refuse to use. Subjective theorists ascribe a subjective meaning to $P(E_1)$, which is employed to qualify *a credence about the occurrence of* E_1. Note how this justification is grounded on semiotic concepts which are amply shared in the literature. The present method of study legitimates the subjective probability with the support of the undeniable semantic properties of $P(E_1)$.

The Bayesian inference uses *priors* $P(\theta)$ which can be determined from past information, such as preceding experiments and even using other techniques. A prior is *informative* when it expresses specific, definite information about a variable; it is *uninformative* or *diffuse* if it expresses vague information about a variable such as "all the outcomes are equally likely." The posterior probability $P(\theta/x)$ is the probability of the parameters θ after the observations of x with likelihood $P(x/\theta)$

$$P(\theta/x) = \frac{P(x/\theta)P(\theta)}{P(x)}.$$

The informational nature of the subjective probability is consistent with the Bayesian logic which employs prior and actual information through appropriate and nonarbitrary criteria.

In conclusion, *Results 1* and *2* provide two distinct answers to *the interpretation problem*, which are derived from TLN, TSN, and semiotics and not from personal opinions.

RESULT 3

The probabilities $P(E_n)$ and $P(E_1)$ qualify the very different events E_n and E_1 that Eqs. (3) and (4) specify this way

$$\begin{aligned} &n \to \infty, \\ &n = 1. \end{aligned} \tag{5}$$

These constraints are disjoined at the point of logic

$$(n > 1) \text{ OR } (n = 1). \tag{6}$$

Therefore, the probability of the long-term event and the probability of a single event do not overlap

$$(n > 1) \text{ OR } (n = 1) \quad \Rightarrow \quad (PE_n) \text{ OR } P(E_1). \tag{7}$$

This statement proves that $P(E_n)$ and $P(E_1)$ occur in different circumstances and are not irreconcilable. The two forms of probability apply to situations that do not interfere.

Several researchers claim that either the frequentist or the subjective model is universal, as long as those researchers overlook assumptions (5) that are typical of E_n and E_1. The present frame shows how $P(E_n)$ and $P(E_1)$ regard specific practical situations, while $P(E)$ has a generic relationship with the physical reality. Each form of applied probability is subjected to special experimental constraints and should not be confused with the abstract form (2). The confusion between applied and abstract forms of probability causes the *discrepancy problem*, whereas mathematical statement (7) proves that this problem does not have a basis.

RESULT 4

Statisticians employ an assortment of mathematical tools; some of these are frequentist, and some are Bayesian. There are Bayesian methods which have a frequentist equivalent and sometimes give the same numerical result. In practice, there are situations in which one of the methods is more preferred by some criteria, while the other method is preferred for other reasons. There are also scholars who believe that each statistics is actually essential for the full development of the other. Bayarri and Berger tell that a certain interplay occurs between classical and Bayesian statistics.[26] Someone even holds a hybrid approach.[27] The theorems of large numbers and a single number take mutually exclusive hypotheses (5), and thus, the statistical methods underpinned by TLN and TSN must be consistent with Eq. (5). The logical divide (6) yields a second divide that is expressed as follows:

If one expert means to investigate a long-term event,
he must resort to using classical statistics;
If one expert means to focus on a single event,
he must adopt Bayesian statistics. (8)

The present frame shows two distinct ways: assumption $n \to \infty$ is consistent with the classical statistics and $n = 1$ is consistent with the Bayesianism, but statements (8) say something more and above. There is no middle way in Eq. (8), and this rule does not allow exceptions. For example, if a doctor is treating a patient who is suffering from cancer Z, he is considering a single case and adopts the Bayesian tools. Instead, if a researcher investigates the epidemiology of cancer Z, he is concerned with a general trend and follows classical statistics. The working statistician has to apply a precise statistical technique depending on the particular question he is dealing with. The two statistical schools work perfectly in practice, provided that they are restricted to a suitable domain of application and rule (8) constitutes the answer to the *Problem of Choice*.

The subjective model underpins the Bayesian statistics, which focuses on single occurrences even if the Bayesian procedures are not confined to a lone observation. When a Bayesian applies to a sequel of repeated events, his conclusions regard each individual case. Results 1 and 2 explain how the significance of the frequency probability does not have anything in common with the subjective meaning, and thus, each personal value $P(E_{x1})$, $P(E_{x2})$, $P(E_{x3})$,… $P(E_{xn})$ cannot be confused with $P(E_n)$ that is an authentic physical parameter.

RESULT 5

The mathematical approach which we are employing leads to the precise organization that divides the abstract calculus from the applied calculus. The probability sector splits into two areas as a consequence of the arguments E and E_n with E_1. The generalized definition of $P(E)$ is unique in accordance with the expectations of Hilbert's sixth problem that proves not to be an ill-posed question (*Problem of axiomatization*). The applied calculus regards special cases such as the long-term event E_n, the single event E_1, even the quantum event $E_{q,}$ the economic event E_x, etc. There are various areas of applications, while the abstract theory is single.

This precise organization of the probability calculus does not falsify the mathematical constructions of the frequentist and subjective authors, rather it shows how those constructs are incomplete. Each theory provides the illustration of the probability calculus under the explicit hypothesis $n \to \infty$ and the hypothesis $n = 1$ in the order, and hence, they offer effective assistance to scientists in each application field but do not provide the exhaustive illustration of the probability concept because of their restricted assumptions.

The reader can note how the mathematical method that we have adopted provide answers to the *Problems* from *A* to *D* which are marked by innovation.

VI. CONCLUSION AND OUTLOOK

The textual analysis provides evidence that opinions and personal choices often influence the works of masters. They not only leave unresolved some foundational issues but also sometimes compound them. Three hundred and fifty years after Pascal's inauguration of the modern probability calculus, there are significant open problems. At present, experts struggle with the *discrepancy problem*, the *choice problem*, and the *axiomatization problem* besides the *interpretation problem*. This failure pushed us to follow the way suggested by Hilbert and inaugurated by Kolmogorov, which centers on mathematics, while verbal explanations serve as completing elements. This approach provides innovative answers to the *Problems A-D*; more precisely,

- the *testability* of $P(E_n)$ and the *unverifiability* of $P(E_1)$ are proved;
- It is explained how $P(E_1)$ *is reused as the subjective probability* instead of being scrapped;
- It is demonstrated why $P(E_n)$ *and* $P(E_1)$ *are not irreconcilable*;
- *Statisticians have a precise rule to follow* when they are called for selecting the most appropriate statistics;
- The probability calculus splits into the abstract calculus of $P(E)$ and the applied calculus of $P(E_n)$ and $P(E_1)$, and hence, *Hilbert's sixth problem is not ill posed*.

Several experts of probability and statistics are inclined to accept both the frequentist and subjective models (see Sec. III), but a rigorous theoretical frame is missing so far. To the best of our knowledge, this inquiry provides the first formal "dualist theory."

Blanshard[28] claims that *philosophy comes before science,* especially when a problem is not well-defined. It is difficult to conceive of a measure when one does not quite know what one is measuring; however, the importance and role of philosophy change with the progress of science. When scientists find the mathematical definition of the searched measure and this equation reaches broad consensus, then the intellectual ruminations progressively slip to the background. For instance, thinkers debated the nature of mechanical force for a long while. When Newton fixed the concept of force with a differential equation, engineers began to calculate mechanical equipment and verbose discussion became unfashionable. The debates about the multifold nature of probability were appropriate in the past since the concept of probability could not be accurately stated and the context of the problem seemed fuzzy. Nowadays, the theorems of large numbers and a single number demonstrate the properties of probability with precision and physical phenomena wait to be calculated using the present results as first in quantum mechanics. In particular, we are going to present a study that sheds new light on the quantum wave collapse.

APPENDIX A: A SHORT REVIEW

1. von Mises devotes the initial part of the third chapter in the book[4] to investigating the opposing definitions of probability; the second part of the third chapter deals with the critical aspects of his theory. von Mises emphasizes the limit of the Laplacian scheme, and in addition he holds that the subjective model may be influenced by psychological or physiological mechanisms, and puts down this model as follows: "The peculiar approach of subjectivists lies in the fact that they consider 'I presume that these cases are equally probable' to be equivalent to 'These cases are equally probable' since for them probability is only a subjective notion."
2. Savage writes three pages in Chap. 4 of *The Foundations of Statistics*[5] with critical remarks on the "objective" interpretations of probability. His view may be summed up with this passage: "In the first place, objectivistic views typically attach probability only to very special events. Thus, on no ordinary objectivistic view would it be meaningful. Secondly objectivistic views are (…) charged with circularity. They are generally predicated on the existence in nature of processes that may (…) be represented by (…) an infinite sequence of independent events."
3. Reichenbach illustrates the historical evolution of the probability concept in the first chapter of Ref. 6. Chapter 9 discusses the various meanings of probability; in particular, he examines the probability of a single event, which should have no place in science. If the concept of probability only represents a subjective expectation, then *P* does not have any connection to the real world.
4. Venn criticizes the interpretation of probability as personal belief—especially in relation to the thought of De Morgan—in Chap. VI of the book.[7] He states: "the difficulty of obtaining any measure of the amount of our belief" and adds: "we experience hope or fear in so many very instances, that (…) whilst we profess to consider the whole quantity of our belief we will in reality consider only a portion of it." Venn concludes that human actual belief is one of the most elusive and variable factors so that we can scarcely ever get sufficiently clear hold of it to measure it. Chapter X tackles another argument; specifically, it questions whether the events calculated with the probability calculus are to be attributed to chance on the one hand or alternatively to causation or design on the other hand.
5. Chapter VII of Keynes's treatise[8] makes a historical retrospect of the probability calculus. He illustrates the

frequency theory recalling the work of Leslie Ellis and mostly looking into the Venn work. Chapter VIII emphasizes the limits of the frequentist model that clearly excludes a great number of judgments which are generally believed to deal with probability. Keynes also stresses the practical use of statistical frequencies since "an event may possess more than one frequency, and that we must decide which of these to prefer on extraneous grounds." Later Keynes emphasizes the differences between Venn's construction and the generalized frequency theory which he means to put forward.

6. de Finetti illustrates his theory in two volumes[9] that are peppered with unfavorable and even sarcastic judgments about the opponent theories. In the first chapter, he places the notions typical of the subjective and objective schools of probability side by side in order to highlight the profound differences extant between them. For instance, he notes that for "objectivists," two events are independent if the occurrence of one does not affect the probability of the other; instead for "subjectivists," two events are independent if the knowledge of one does not modify the assessment of the probability of the other event. Chapter VI introduces three main interpretations of distributions, and then, the author begins a long discussion against countable additivity. Chapter XII deals with estimations and testing that have distinct characters from the perspectives of the classical and Bayesian statistics. de Finetti never fails to emphasize the Bayesian techniques and to criticize the alternative methods.
7. Ramsey begins the seventh chapter of the essay[10] with censorious comments on the works of von Mises and Keynes. He pinpoints that the latter recognizes the subjectivity of probability, but in substances, Keynes does not assign any value to subjectivism. Moreover, Keynes believes there is an objective relationship between knowledge and probability, as knowledge is disembodied and not personal. Ramsey analyses the connection between the subjective degrees of belief an individual has in a proposition and the probability it can be given. As regards the frequentist theory he writes: "I am willing for the present to concede to the frequency theory that probability as used in modern science is really the same as frequency."

APPENDIX B: UPPER-BOUND LEMMA

TSN proves that probability is unreal when the argument is a single event. We could say that $n = 1$ is the lower bound of the probability inexistence. Let us examine the largest number of events whose probability is unreal.

Lemma. Suppose z is any positive integer, the probability of E verifies

$$P(E) = 1/z, \quad z > 0. \tag{B1}$$

Then, the relative frequency of the successful event E in n trials is not equal to the probability

$$F(E_n) \neq P(E). \tag{B2}$$

If

$$1 < n < z. \tag{B3}$$

Proof. We proceed by absurd and deny (B2), and we put the relative frequency $F(E_n) = N(E_n)/n$ equal to the probability

$$N(E_n)/n = P(E) \tag{B4}$$

when the event E occurs one time

$$N(E_n) = 1.$$

We obtain from Eqs. (B4) and (B1)

$$1/n = 1/z.$$

Thus,

$$n = z.$$

This conclusion mismatches with assumption (B3), and therefore, Eq. (B4) is false and Eq. (B2) is true.

Example. The probability of getting a king from a card deck

$$P(E_K) = 4/52 = 1/13 = 1/z.$$

If the number of trials is less than thirteen

$$13 > n > 1. \tag{B5}$$

There are two possibilities. If one does not get any king in n drawings, the relative frequency is lower than $P(E_K)$

$$0/n < 1/13$$

If one gets one (or more) kings in n experiments, the relative frequency is greater than $P(E_K)$. Suppose minimizing the number of successes and maximizing the number of drawings, we get

$$1/12 > 1/13.$$

In summary, the relative frequency never collides with the probability $P(E_K)$ that proves to be out-of-control and therefore can express a personal degree of belief in relation to the condition (B5).

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